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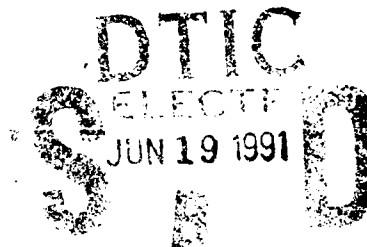
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Fitting Seasonal Averages with a Continuous Function



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Foreword

Ocean circulation models have the potential to provide accurate forecasts, as well as nowcasts. The development of these models is facilitated by the existence of continuous time series of quantities to which they can be compared. The Levitus climatology provides one standard, but most useful quantities are available only as seasonal means. Several algorithms are presented to convert these into the continuous form that is required.



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Executive Summary

A large-scale, integrated program in ocean circulation model development has been pursued by the Ocean Modeling and Prediction Branch, Ocean Sensing and Prediction Division, Naval Ocean Research and Development Activity. This program will provide the Navy with accurate nowcasts and forecasts of the state of the ocean on global, regional and tactical scales. One of the constraints that has been applied to this research is the development of circulation models that accurately reproduce the measured climatologies of such quantities as the dynamic height, thermocline depth, and depth-averaged density. These fields are derived using in situ data that have been acquired over many decades, but are so irregularly distributed that in most regions only annual or seasonal mean fields have been derived. Comparisons between model and data climatologies are facilitated if both are available as continuous time series, but the standard Levitus climatology is generally available at time scales no shorter than seasonal. This report describes several methods for converting the existing seasonal quantities into a continuous time series of fields, and also notes an amplitude error and bias present in the seasonal values as presently derived.

This work was supported by the Naval Ocean Modeling Program, Mr. Robert Peloquin, Program Manager, under Program Element 62435N. We thank the NOMP program for its support. In addition, we would like to extend our appreciation to Dr. Harley Hurlburt (NORDA 323) for several useful discussions.

The mention of commercial products or the use of company names does not in any way imply endorsement by the U.S. Navy or NORDA.

Acknowledgments

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Fitting Seasonal Averages with a Continuous Function

Overview

The Levitus climatology contains annual and seasonal averages for potential temperature, salinity, and density (Levitus, 1982). To include this information in an ocean model (either as a forcing function or a climatological relaxation term), it is desirable to compute a more continuous version of these quantities. In the past, these data have simply been interpolated, but this process ignores the filtering effects created by the seasonal averaging process. The basic philosophy of all the methods described here is to develop a continuous function whose seasonal averages match those given by the climatology.

The first part of this report describes a basic problem with the raw data (a bias resulting from the seasonal averaging and sampling process) and some solutions. The remainder of the report describes various methods for generating a continuous function to represent the climatology. Fourier methods, polynomial interpolation, and splines of various orders are analyzed and compared. The methods produced similar continuous function, as well as similar first derivatives. The second derivative in the case of the polynomial and spline forms is discontinuous between years or seasons, and may be of concern.

Aliasing Present in the Climatologies

We can simulate the process of developing a set of seasonal averages and can show that there are two basic problems. Seasonal averaging involves taking all the data that occurs in a particular season and averaging it. The Levitus climatologies are computed in this way. Mathematically, this process is equivalent to convolving the original "continuous" time series with a 91-day "running average," then sampling the resulting smooth time series at a 91-day increment. The averaging filter significantly reduces the amplitudes of Fourier components whose periods are shorter than 91 days, but by no means entirely removes them. Figures 1 and 2 show the results of computing seasonal time series from sine and cosine waves of various periods. When this averaged time series is resampled at a seasonal (91-day) increment, all the energy that remains at scales shorter than 182 days is aliased into the longer scale components. Figures 3 and 4 show the response of the seasonal averaging filter. All energy with time scales shorter than 182 days are reduced in amplitude by the 91-day running average. When these data are sampled at a 91-day increment, any energy with scales shorter than 182 days is aliased into longer period components. Once this aliasing has occurred, the process cannot be reversed. The seasonally averaged quantities are corrupted to the degree that significant energy at scales shorter than 91 days exists in the real ocean. The red spectra of many oceanographic quantities generally imply that the shorter the time scale, the weaker the phenomenon (at least until the scale is 1 day, at which point the tidal energy becomes significant). Exceptions occur at inertial and tidal periods.

Potential Undersampling Problems

The red spectrum can be interpreted conversely, as well: the longer the time scale, the stronger the phenomenon. Both real, in situ data and model simulations show that the sea surface height (SSH) and the subthermocline pressure field have their strongest components at scales much longer than 1 year. Figure 5 displays the spectrum of sea surface height as measured by

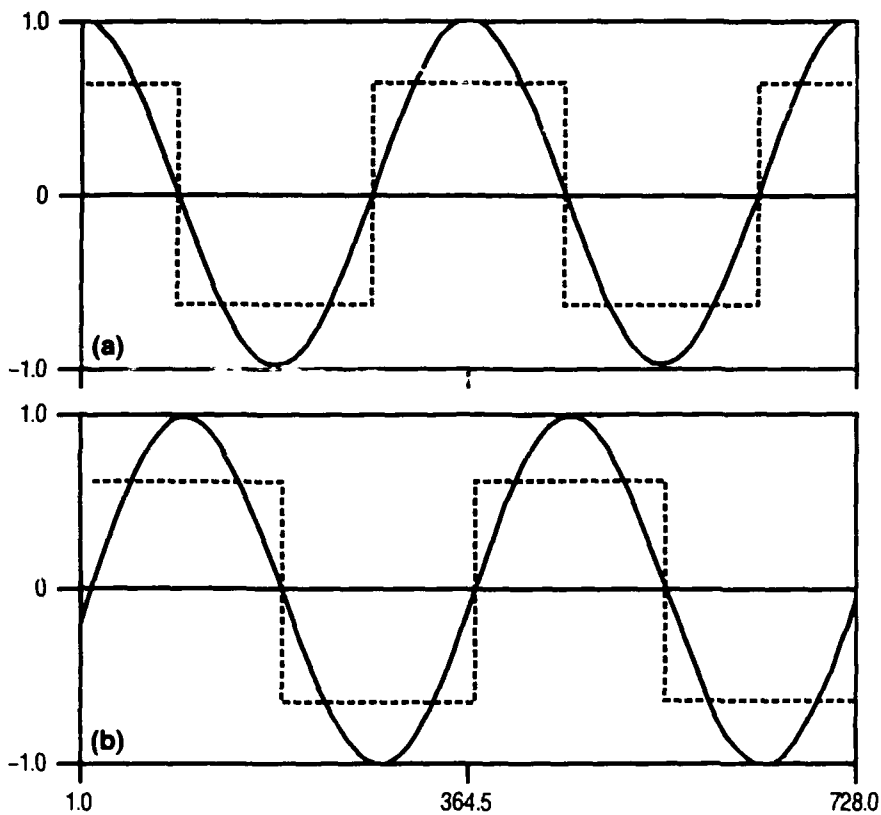


Figure 1. The basic cosine and sine series used to build the continuous estimate $b(t)$ to the seasonal values $(b_1 \dots b_4)$. This figure shows the results of taking 2 years of sine and cosine waves with a period of a year and reducing them to seasonal means.

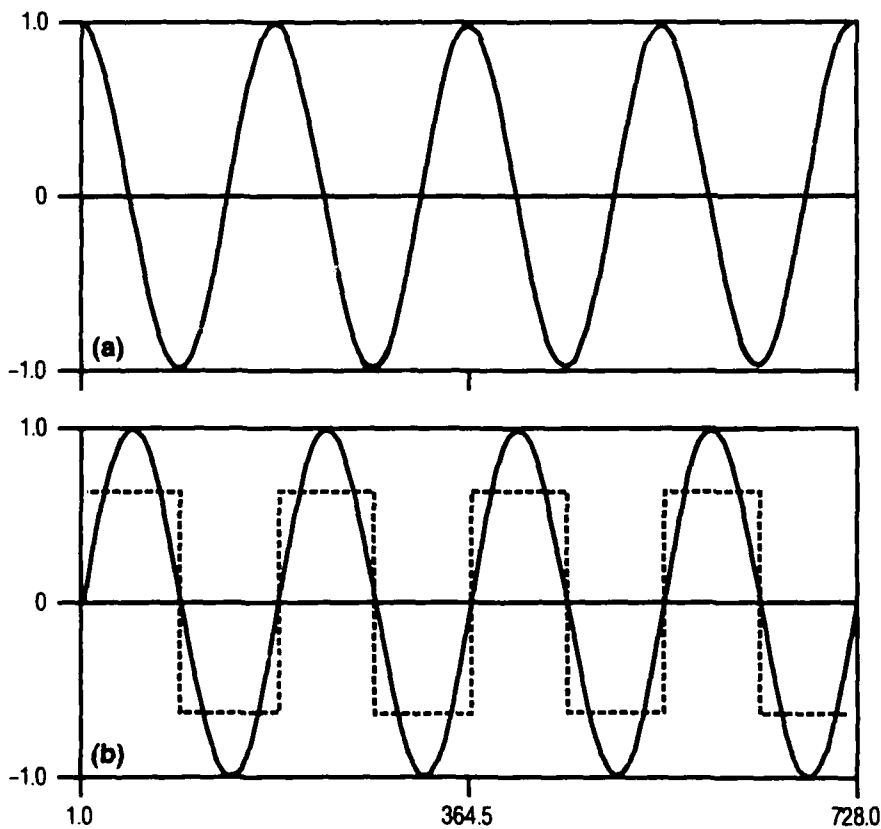


Figure 2. Same as Figure 1, except that the basic period is only a half-year long. The cosine component is completely removed by the process of seasonal averaging.

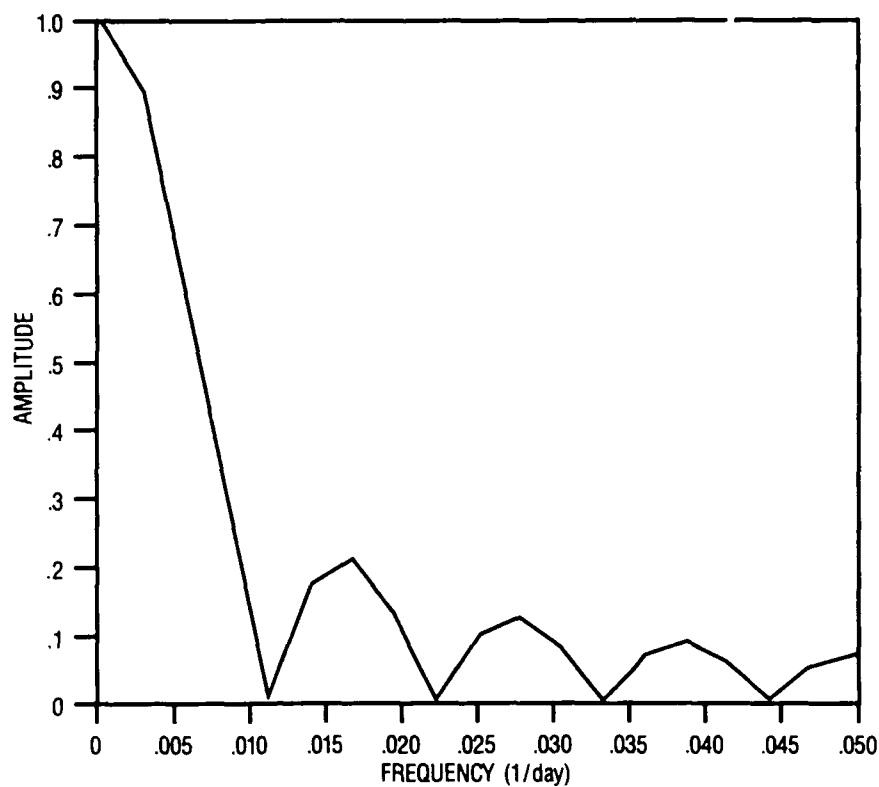


Figure 3. The response function of a 91-day running average (as a function of frequency in units of 1/day).

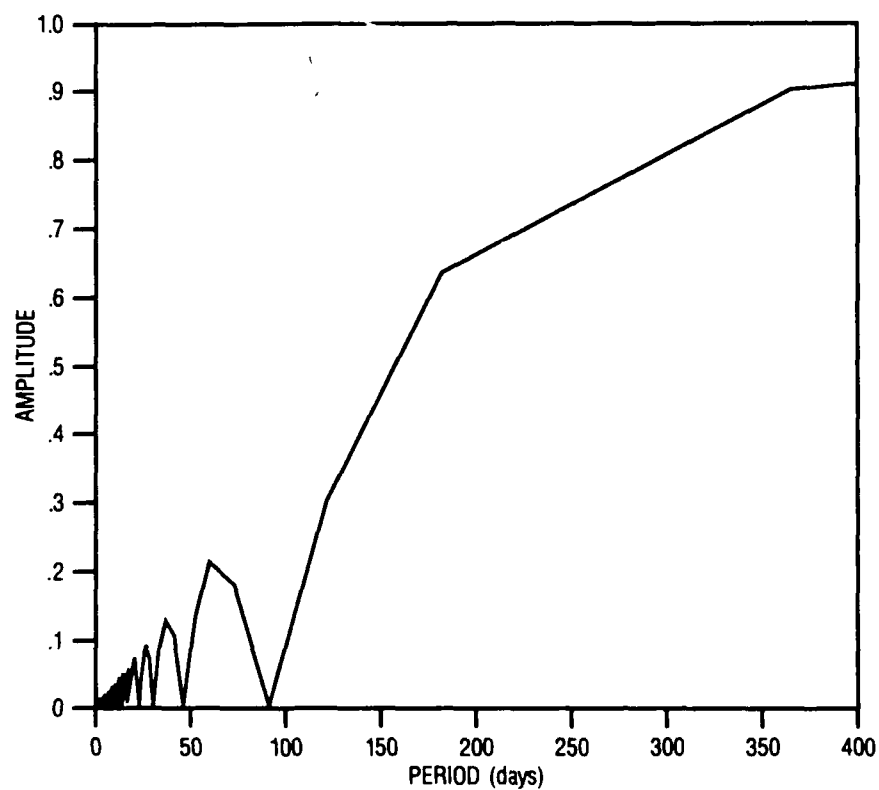


Figure 4. Same as Figure 3, except displayed as a function of period in days. Energy at scales shorter than 91 days is significant. Any periodicities in the actual data that are also shorter than 91 days will be aliased into longer periods when reduced to seasonal means.

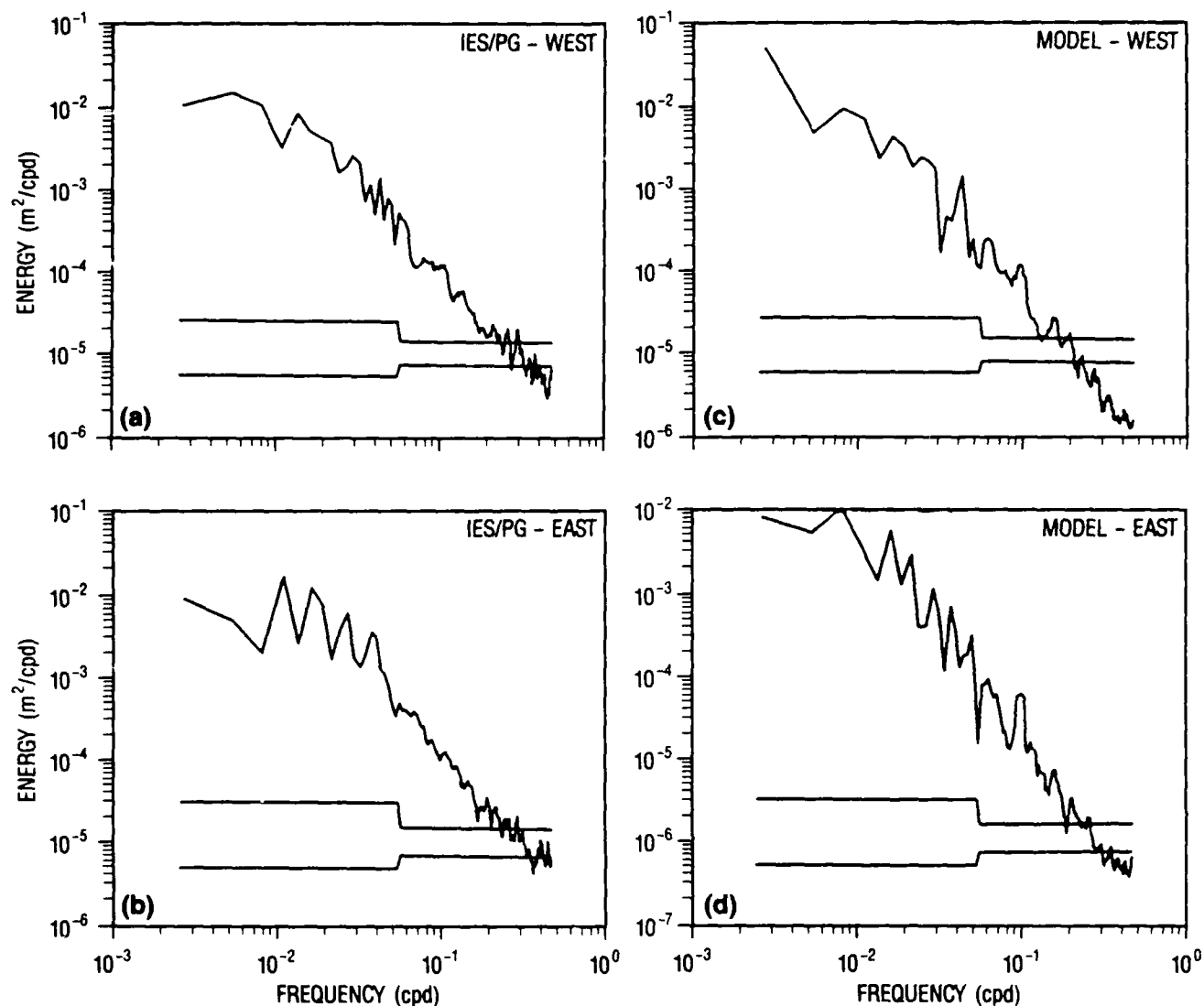


Figure 5. Spectra of total SSH anomaly over a 380-day period comparing IES/PG in situ data and model-simulated data at two locations. Both data sets show significantly red character.

an Inverted Echo Sounder (IES) array (Fox et al., 1988) in the North Atlantic, compared to an ocean model result. Extremely long model simulations show that significant energy exists at scales as long as 10 years (Fig. 6). Any climatology developed using only a few years of data will gradually become obsolete, as longer-term components treated as constant begin to change. Climatologies based on 1-year averages will be outmoded rapidly since there is significant variability on time scales longer than 1 year. Figure 7 displays climatologies based on 1-year averages of SSH taken from a model simulation of the North Atlantic at various locations compared to a climatology based on all 10 years of the same data. In this simulation, interannual variations in the annual mean sea surface height are clearly substantial. The Levitus climatology is based on between 10 and 30 years of data, but no data from the past 10 years have been included. If the variability in the measured quantities is significant on scales longer than 10 years, then this climatology may already be out of date for some purposes.

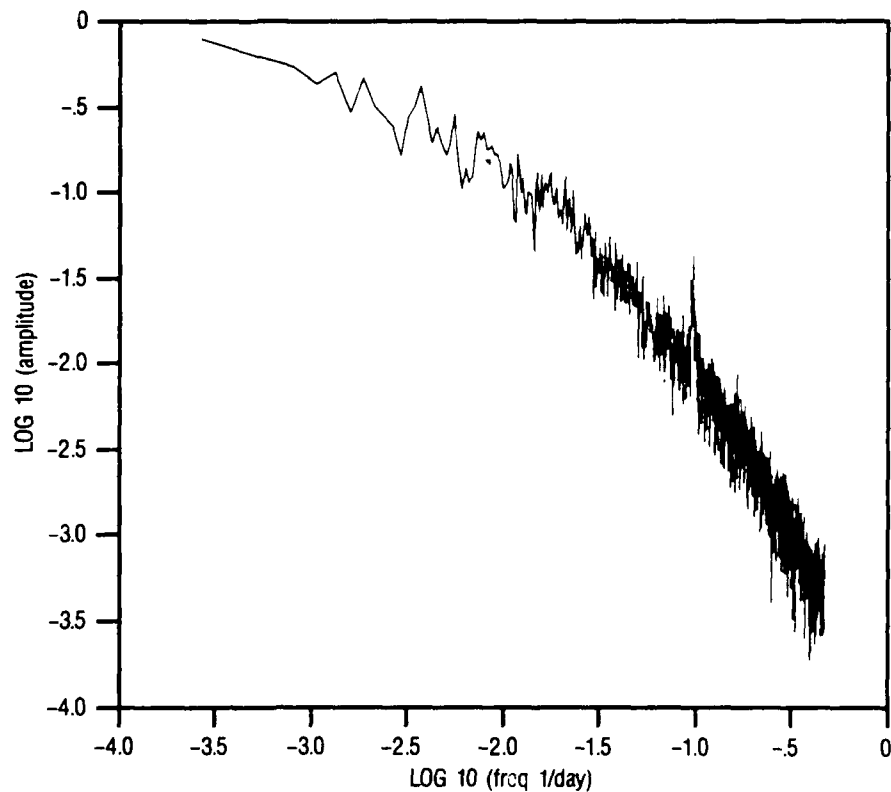


Figure 6. Spectrum of SSH for a 10-year model simulation of the North Atlantic showing that the redness of the spectrum extends to scales of many years.

Biasing of the Semiannual Component

After smoothing and resampling the continuous time series to form seasonal averages, the remaining Fourier components are the annual and semiannual periods. Figure 2 shows the effects on the semiannual component of averaging and resampling to seasonal averages. Using January 1 as the start of the time axis ($t = 0$), the semiannual sine component is clearly reduced, but the cosine component is completely filtered out. Even if the original time series consisted of only annual and semiannual components, converting this data into seasonal averages destroys the semiannual cosine wave, so that the original simple time series cannot be reconstructed. The sine component can be computed, but not the cosine. In the Fourier methods given here, the annual and semiannual components are computed, but it should be emphasized that the semiannual component is biased in that its phase is always 90° (always a pure sine component, with no cosine component). The potential effects are subtle and not-so-subtle. For example, whatever energy exists in the real ocean at semiannual scales will be partially reduced, since only its sine component can be estimated. As an extreme example, consider deriving a continuous wind series to drive the model for seasonal winds. The driving at semiannual time scales will be weaker than it should be (on average, about 71% of what the total forcing should be). An additional effect of this bias will be that the semiannual forcing will always be at a particular global phase. A true, continuous climatology would almost surely have a semiannual phase that varies with location.

Options for dealing with this problem range from doing nothing to attempting to derive the semiannual phase relationship in some way. Rather than include only the sine component of the semiannual climatology, the semiannual variability could be totally ignored and only the annual component used. In some areas of the world (i.e., the Indian Ocean) the semiannual component is very strong and using only an annual component of the climatology would significantly reduce the "information energy"

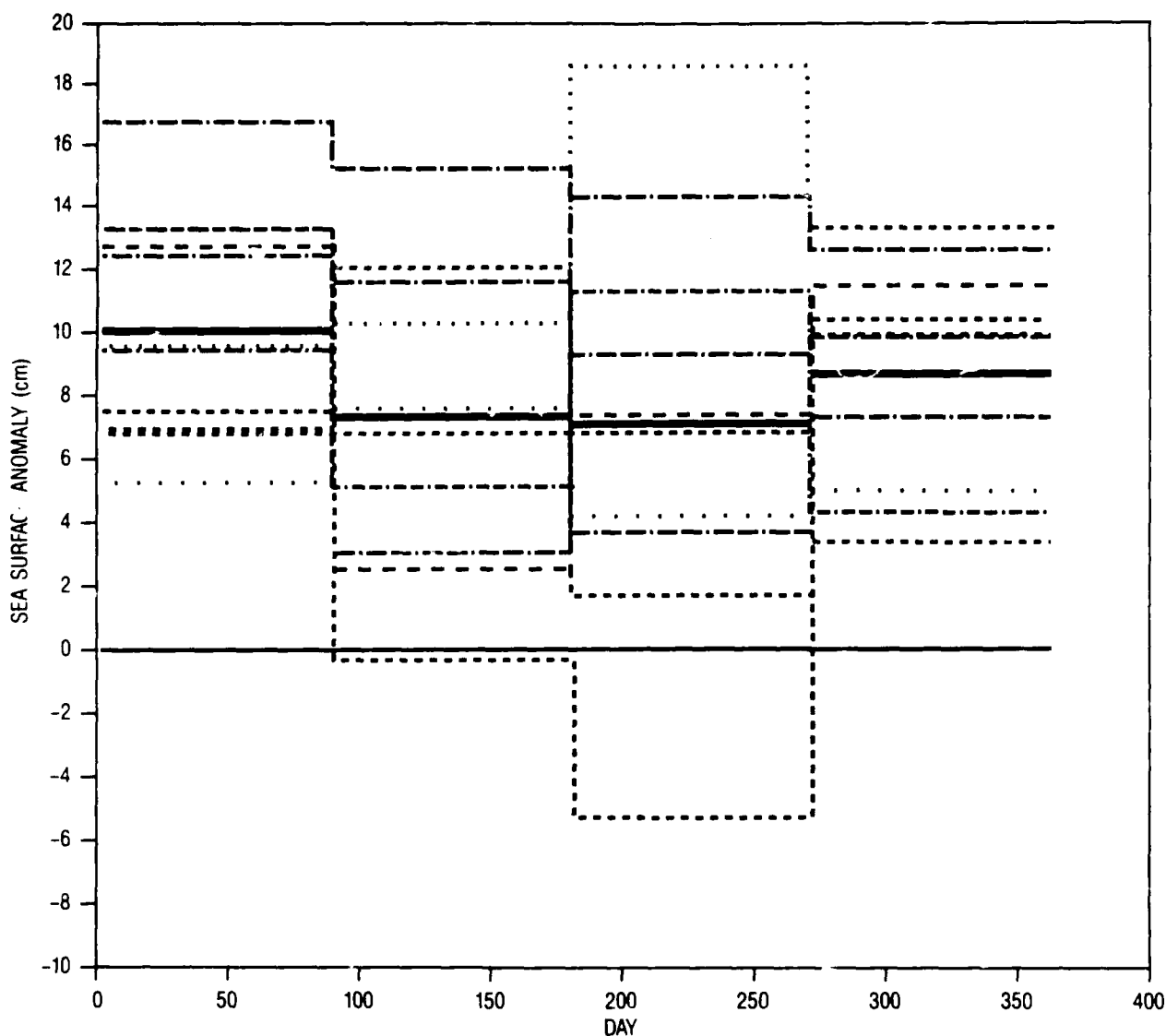


Figure 7. A comparison of the seasonal climatologies formed from the 10-year-long data set of Figure 6 and similar climatologies formed from individual years of the same data set. Interannual variations in the climatology are as large as 10 cm in this data set.

being supplied to the model. The problem of transitioning the model from being driven by climatological data (to spin it up) to driving it with real and current information will always be present. It is probably more important to include the most information possible, regardless of bias, but this remains to be tested and proven.

Three distinctly different methods were used to develop continuous functions that represent the given data (which consists simply of four seasonal averages at any given geographical location). In each case, a set of best-fit coefficients, which represent Fourier coefficient weights or terms in some polynomial equation, are derived. Each grid point in the Levitus climatology is assumed to be separately treated, which might result in some undesired spatial variability (unverified). If this is the case, the coefficients could be spatially smoothed before being used to create the continuous interpolating functions.

A series of random seasonal climatologies were generated and fitted with each of the methods described. Three samples are shown in Figures 8

Overview of Interpolation Methods

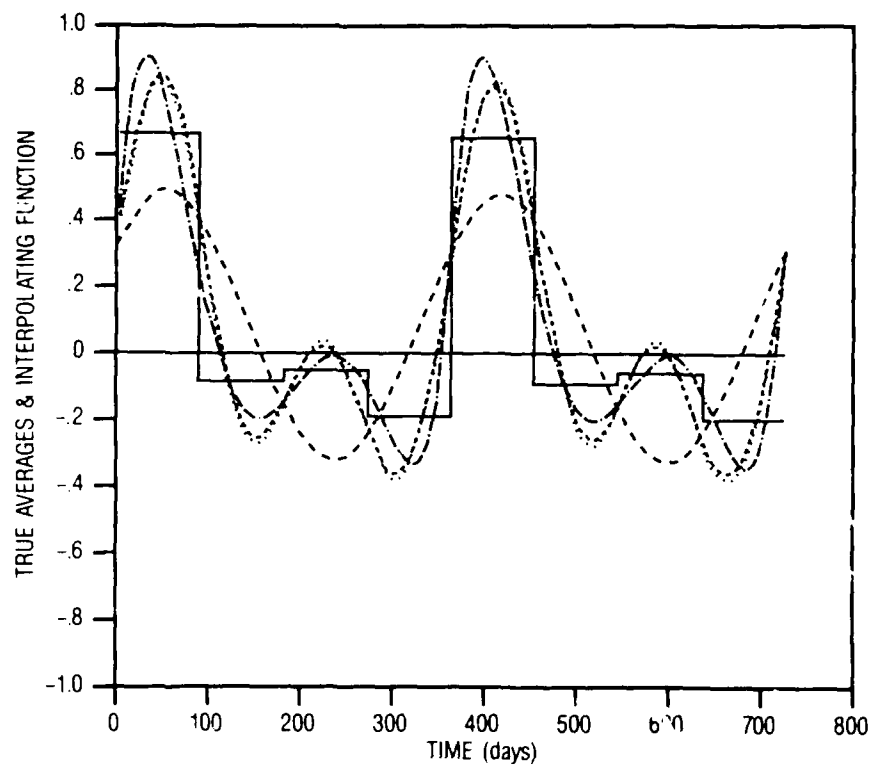
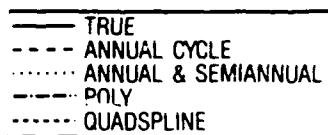


Figure 8. Two years of synthetic seasonal values after being converted into continuous time series by the methods described in this report.

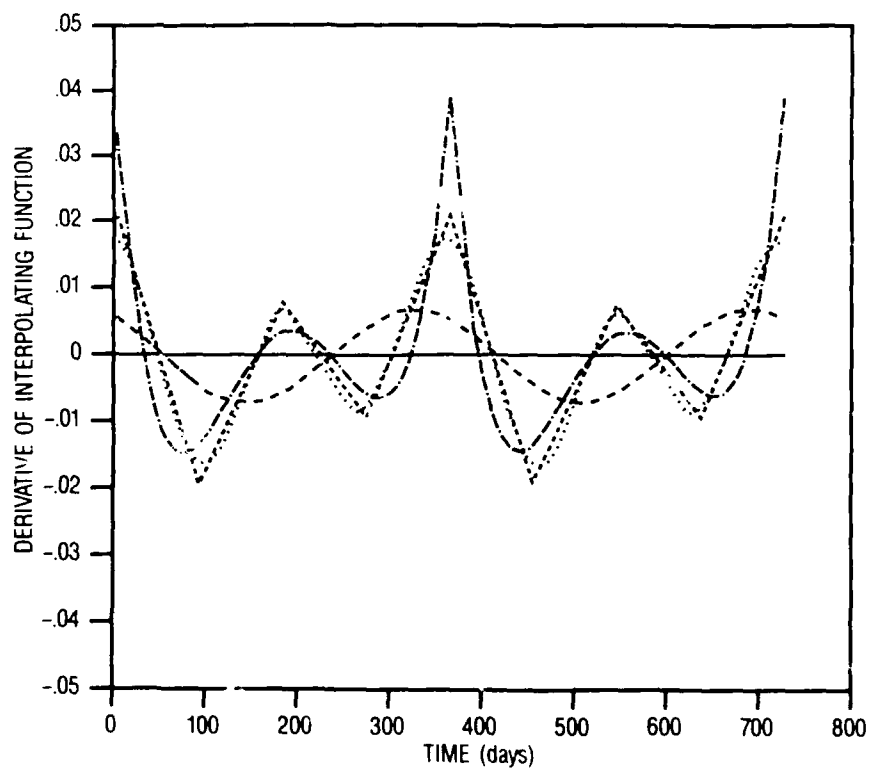
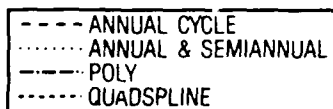


Figure 9. The instantaneous time derivative of the curves shown in Figure 8.

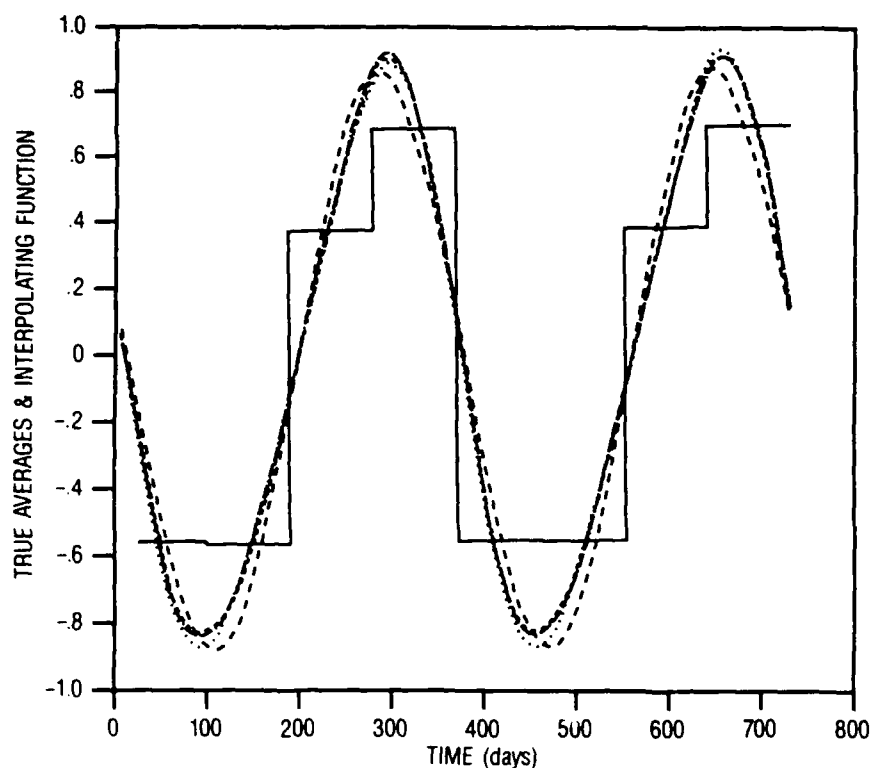


Figure 10. Another realization
(see Fig. 8).

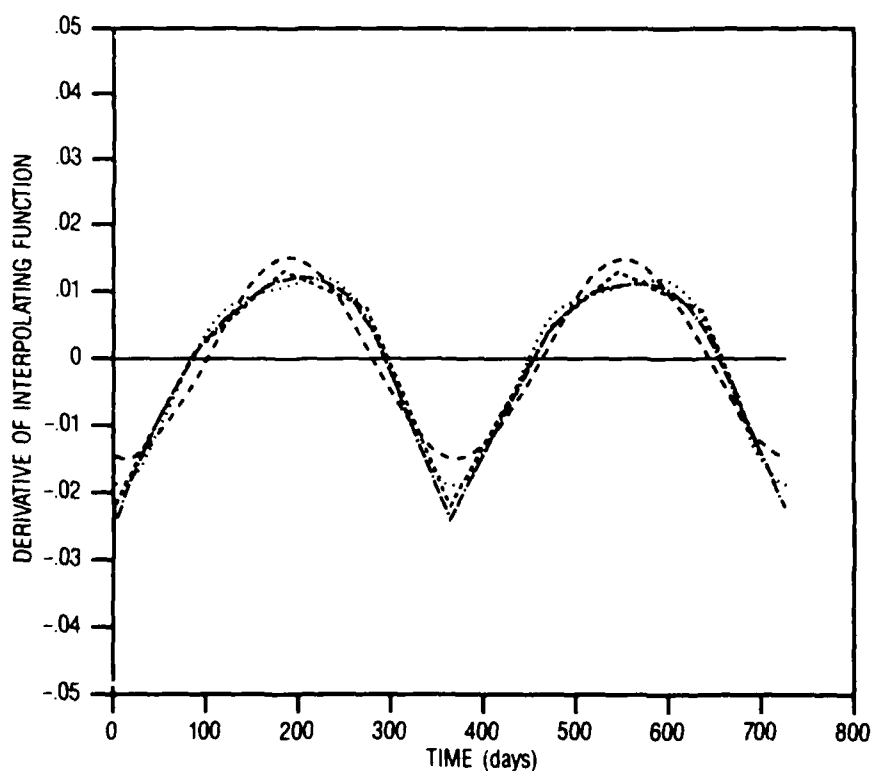


Figure 11. Another realization
(see Fig. 9).

through 13. In each plot, 2 consecutive years of identical data are shown so that the continuity across the annual boundary is evident. The "square wave" solid line represents the seasonal average drawn to cover its entire season. Each example consists of a plot of the data converted into continuous form by each of the methods described in the following text, followed by

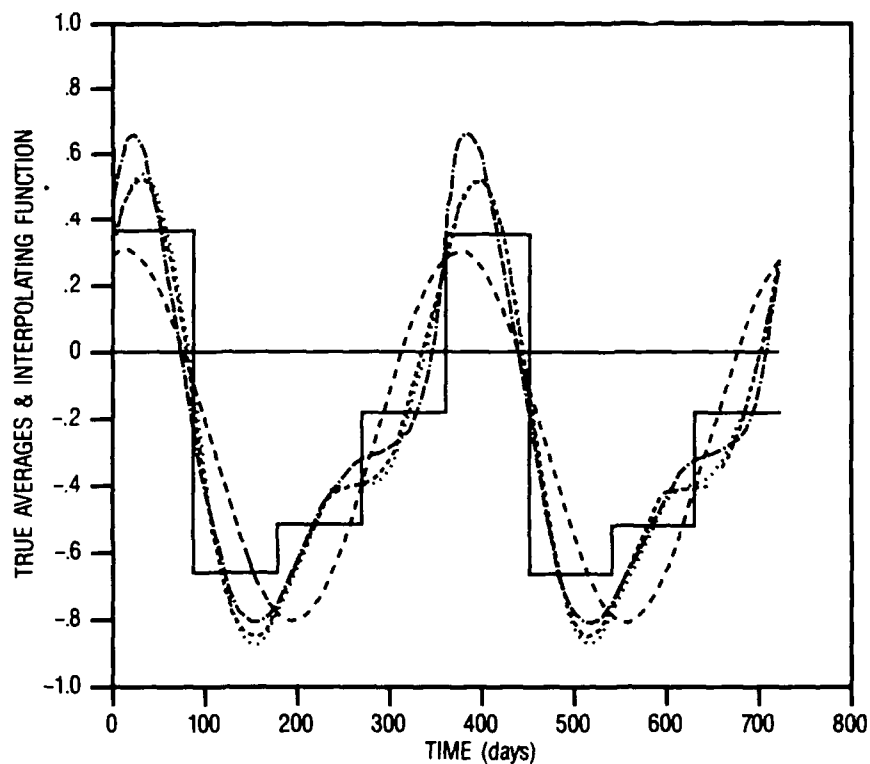
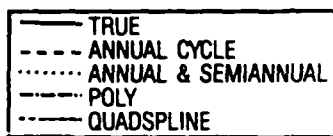


Figure 12. Another realization (see Fig. 8).

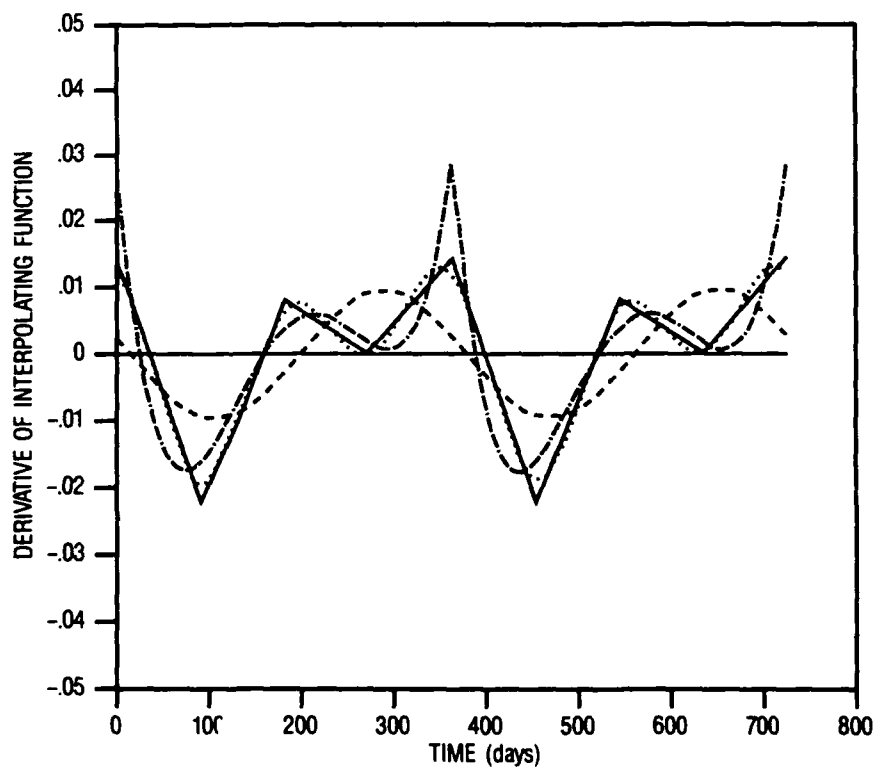
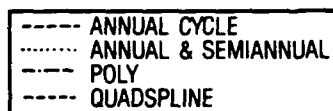


Figure 13. Another realization (see Fig. 9).

a plot of the time *derivative* of the continuous interpolating time series. The three methods yield very similar time series and similar derivatives. This similarity is emphasized in Table 1, which summarizes the results of 35 experiments (similar to results displayed in Figs. 8 through 13). In each case, the variability of the interpolated continuous time series is computed for

Annual Only	Annual and Semiannual	5th Degree Polynomial	Quadratic Splines	Mean Variability	STD Variability	Percent Deviation
0.1061	0.1866	0.1774	0.1815	0.1818	0.0038	2.0748
0.3411	0.3473	0.3372	0.3378	0.3408	0.0046	1.3550
0.1851	0.2282	0.2292	0.2220	0.2265	0.0032	1.4091
0.1463	0.2057	0.1901	0.2001	0.1987	0.0065	3.2471
0.0697	0.2903	0.3203	0.2824	0.2977	0.0163	5.4776
0.1928	0.2128	0.2322	0.2069	0.2173	0.0108	4.9684
0.2352	0.2544	0.2768	0.2474	0.2595	0.0125	4.8280
0.2358	0.2472	0.2639	0.2404	0.2505	0.0099	3.9443
0.1856	0.2262	0.2411	0.2200	0.2291	0.0088	3.8490
0.1573	0.1984	0.1850	0.1930	0.1921	0.0055	2.8672
0.2385	0.2562	0.2665	0.2492	0.2573	0.0071	2.7656
0.0868	0.2166	0.2630	0.2107	0.2301	0.0234	10.1643
0.0257	0.3446	0.3857	0.3352	0.3552	0.0219	6.1734
0.1565	0.2350	0.2156	0.2286	0.2264	0.0081	3.5588
0.0937	0.1958	0.1963	0.1904	0.1942	0.0026	1.3579
0.2023	0.2201	0.2324	0.2140	0.2222	0.0077	3.4493
0.1632	0.1931	0.1866	0.1878	0.1892	0.0028	1.4968
0.1862	0.2536	0.2611	0.2467	0.2538	0.0059	2.3254
0.1875	0.2769	0.2906	0.2694	0.2790	0.0088	3.1545
0.1180	0.1827	0.2109	0.1777	0.1905	0.0146	7.6534
0.2983	0.3149	0.2991	0.3063	0.3068	0.0065	2.1050
0.0649	0.2277	0.2386	0.2214	0.2292	0.0071	3.0901
0.1088	0.2394	0.2638	0.2328	0.2453	0.0133	5.4248
0.2701	0.2704	0.2747	0.2630	0.2694	0.0048	1.7839
0.0987	0.2404	0.2665	0.2338	0.2469	0.0141	5.7158
0.3058	0.3269	0.3316	0.3180	0.3255	0.0057	1.7361
0.0760	0.2453	0.2760	0.2386	0.2533	0.0163	6.4349
0.1113	0.2253	0.2726	0.2192	0.2390	0.0239	9.9833
0.2134	0.2236	0.2155	0.2175	0.2188	0.0035	1.5795
0.0559	0.2287	0.2384	0.2224	0.2299	0.0066	2.8630
0.2210	0.2278	0.2393	0.2215	0.2295	0.0074	3.2075
0.2065	0.2196	0.2158	0.2136	0.2163	0.0025	1.1481
0.0302	0.3235	0.3899	0.3147	0.3427	0.0336	9.7935
0.2471	0.2516	0.2456	0.2447	0.2473	0.0030	1.2290
0.3604	0.3701	0.3568	0.3600	0.3623	0.0057	1.5628

Table 1. Thirty-five realizations of experiments (see Figs. 8-13) comparing variabilities of the continuous time series constructed by the various methods described in the text. The annual-only method is excluded in the final three columns.

each method and then intercompared. Excluding the fit based on the annual cycle alone, the time series created have variabilities that deviate from one another by no more than 10%, and the bulk of the experiments showed differences of only a few percent.

The first method described is based on developing a Fourier series that represents either the annual component alone, or both the annual and semiannual waves. As described, when the semiannual component is included in the fit, there is a bias in that only the sine component can be estimated from the data. However, if only the annual component is used, significant energy present in the semiannual waves is ignored. In both cases, the interpolating function is continuous throughout the year, as well as from year to year, and so are all its derivatives. This continuity is important if the model being driven directly or indirectly by the climatology responds to temporal derivatives of the quantities being supplied. In the samples (Figs. 8 through 13), the annual-only fit is shown with a long dash line and the annual plus semiannual fit is shown with a dotted line.

The second method is based on polynomial interpolation. We attempt to derive a finite polynomial which, when averaged, yields the

correct seasonal values. The polynomial and its derivatives are continuous throughout the year, but it is shown that at the year-to-year boundaries, continuity constraints must be relaxed for a solution to exist. For example, in the case of a 5th degree polynomial, we can require continuity of f and f' , but in the case of a 6th degree polynomial, we cannot add continuity of f'' and still find a solution. The 5th degree polynomial fit is displayed in Figures 8 through 13 using a dot dashed line.

The final method of solution is based on splines of various degrees. Each season is represented by a separate polynomial (linear, quadratic, or cubic) and then boundary conditions of continuity are used to fix the terms. It is shown that linear and cubic splines are not uniquely determined by the equations and that some additional ad hoc constraint must be imposed. Quadratic splines can be derived without problem (indicated in Figs. 8 through 13 by a short dash line).

In each of the methods described below, the four known seasonal average quantities are represented by the symbols b_1 through b_4 .

Fourier Interpolation

Assuming that the underlying time series had a Fourier representation, the seasonal averaging process will produce a time series that has a dc component, a cosine and sine component with an annual cycle, and a sine component with a semiannual cycle. Any other frequencies are aliased or averaged out. The most general Fourier expansion we can write then for the original time series is

$$b(t) = a_0 + a_1 \cos(\omega t) + a_2 \sin(\omega t) + a_3 \sin(2\omega t), \quad (1)$$

where ω is the frequency corresponding to the annual cycle. If time is measured in days, then the first day (Jan 1) corresponds to $t = 0$ and the fundamental frequency will be given by $\omega = 2\pi/364$ (for the case of a model year with 364 days). If time is measured in weeks, then $\omega = 2\pi/52$ and the first week will be at $t = 0$.

The four seasonal averages of Equation (1) are given by:

$$b_1 = \langle b(t) \rangle_1 = a_0 + a_1 \langle \cos(\omega t) \rangle_1 + \dots + a_3 \langle \sin(2\omega t) \rangle_1, \quad (2)$$

$$b_2 = \langle b(t) \rangle_2 = a_0 + a_1 \langle \cos(\omega t) \rangle_2 + \dots + a_3 \langle \sin(2\omega t) \rangle_2, \quad (3)$$

$$b_3 = \langle b(t) \rangle_3 = a_0 + a_1 \langle \cos(\omega t) \rangle_3 + \dots + a_3 \langle \sin(2\omega t) \rangle_3, \quad (4)$$

$$b_4 = \langle b(t) \rangle_4 = a_0 + a_1 \langle \cos(\omega t) \rangle_4 + \dots + a_3 \langle \sin(2\omega t) \rangle_4, \quad (5)$$

where b_1 represents the January-March average, b_2 the April-June average, etc., and the symbol $\langle \dots \rangle_i$ means the average for season i . Performing the various averages as continuous integrations of the sinusoids provides the key to the simplicity of the final algorithm: all the various averages reduce to either $+2/\pi$ or $-2/\pi$.

The final seasonal average equations are, simply,

$$b_1 = a_0 + (2/\pi) (+ a_1 + a_2 + a_3), \quad (6)$$

$$b_2 = a_0 + (2/\pi) (- a_1 + a_2 - a_3), \quad (7)$$

$$b_3 = a_0 + (2/\pi) (- a_1 - a_2 + a_3), \quad (8)$$

$$b_4 = a_0 + (2/\pi) (+ a_1 - a_2 - a_3). \quad (9)$$

Adding all four equations reproduces the expected value for the constant term,

$$a_0 = (b_1 + b_2 + b_3 + b_4). \quad (10)$$

Adding the resulting equations in various pairs produces the final estimates for the remaining a 's:

$$a_1 = (\pi/8) (b_1 - b_2 - b_3 + b_4), \quad (11)$$

$$a_2 = (\pi/8) (b_1 + b_2 - b_3 - b_4), \quad (12)$$

$$a_3 = (\pi/8) (b_1 - b_2 + b_3 - b_4). \quad (13)$$

Thus, from the original seasonal amplitudes b_i , we can immediately construct the coefficients a_i in the expansion. Seasonally averaging the time series will reproduce the given b_i exactly.

If we choose to ignore the semiannual component, we can use the fact that this component is orthogonal to the annual one, so that simply setting $a_3 = 0$ will give the least-squares best fit. Alternately, we can go through the mechanics and verify that the constant and annual cosine and sine terms have the same coefficients as derived above.

One obvious way to interpolate the given seasonal averages is to use a finite polynomial. That is, we attempt to find a polynomial of degree N such that the time series

Polynomial Interpolation

$$b(t) = \sum_{i=0}^N c_i t^i \quad (14)$$

integrated over 3-month periods provides the proper seasonal averages. The additional constraint in the problem, however, is that the time series must have a period of 1 year, and it should be continuous (and have continuous derivatives) everywhere, including where the function shifts from the end of the year to the beginning of the next year. If we normalize the time variable to the range $[0,1]$, then we have the additional set of N constraints:

$$b(0) = b(1), \quad (15)$$

$$b'(0) = b'(1), \quad (16)$$

$$b''(0) = b''(1), \quad (17)$$

$$\dots, \quad (18)$$

$$b^{(N-1)}(0) = b^{(N-1)}(1) . \quad (19)$$

A final constraint,

$$b^{(N)}(0) = b^{(N)}(1) , \quad (20)$$

is satisfied automatically because, it simply requires that $c_N = c_N$, but problems are created. We have $N + 1$ unknowns (c_i), but $N + 4$ constraints (the N equations in (20), plus the four seasonal averages that are required). One might be tempted to solve this as a least-squares problem, but first it will be shown that the derivative constraints prevent a solution from being found.

As an example, take the simple quadratic function,

$$f(t) = a + bt + ct^2 . \quad (21)$$

The time variable is normalized to go from $t = 0$ to $t = 1$, so the requirement of continuity of the function and its derivatives implies the following set of equations (note first that $f'(t) = b + 2ct$):

$$f(0) = f(1) \rightarrow a = a + b + c , \quad (22)$$

$$f'(0) = f'(1) \rightarrow b = b + 2c . \quad (23)$$

The last equation sets c to zero. The previous equation then requires that b be zero as well. Generally, one can show that the only *finite* polynomial function, which is periodic and continuous with continuous derivatives that match across the end points, is the function $f(t) = \text{constant}$. (If we want an *infinite* polynomial, all we have to do is expand the Fourier series derived above for the annual and semiannual fit.)

Polynomial Interpolation with Relaxed Constraints

As seen in the last section, no single, finite polynomial will interpolate the seasonal data in a satisfactory way. The constraint of having all its derivatives continuous across yearly boundaries would have to be removed to permit a solution. A discontinuous derivative in the time domain implies a significant amount of power at high frequencies in the Fourier domain, which is undesirable, but we can still generate the polynomial and assess its accuracy.

We have four constraint equations that come from the given seasonal averages, and we probably want the function and at least its first derivative to be continuous across yearly boundaries. This operation provides six constraints, so the simplest polynomial will have the form,

$$b(t) = \sum_{i=0}^5 c_i t^i . \quad (24)$$

It is convenient to define the time to be in the range $[-2,2]$. That is, $t = -2$ represents the beginning of January 1 and $t = +2$ would be the end of December 31. Integrating $b(t)$ from $t = -2$ to $t = -1$ yields the average for the first 3 months of the year, which should equal b_1 ; integrating from $t = -1$ to $t = 0$ should yield the given average for the second 3 months of the year b_2 , and so on. Continuity of $b(t)$ and $b'(t)$ yields the final two constraints. These six equations in the six unknowns are represented by the matrix equation,

$$\begin{pmatrix} 1 & -3/2 & 7/3 & -15/4 & 31/5 & -21/2 & 127/7 \\ 1 & -1/2 & 1/3 & -1/4 & 1/5 & -1/6 & 1/7 \\ 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1 & 3/2 & 7/3 & 15/4 & 31/5 & 21/2 & 127/7 \\ 0 & 1 & 0 & 4 & 0 & 16 & 0 \\ 0 & 0 & 1 & 0 & 8 & 0 & 48 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \\ 0 \end{pmatrix} \quad (25)$$

This matrix can be inverted exactly to yield:

$$c_0 = (-37/300)(b_1 + b_4) + (187/300)(b_2 + b_3), \quad (26)$$

$$c_1 = (5/18)(b_1 - b_4) - (3/2)(b_2 - b_3), \quad (27)$$

$$c_2 = (2/5)(b_1 - b_2 - b_3 + b_4), \quad (28)$$

$$c_3 = (-47/72)(b_1 - b_4) + (9/8)(b_2 - b_3), \quad (29)$$

$$c_4 = (-1/20)(b_1 - b_2 - b_3 + b_4), \quad (30)$$

$$c_5 = (7/48)(b_1 - b_4) - (3/16)(b_2 - b_3). \quad (31)$$

At each grid point in the model, then, the four seasonal average quantities are transformed into a set of six coefficients for a 5th degree interpolating polynomial.

In the normal method of spline interpolation, we are given a set of data points, and we desire a set of cubic polynomials defined between the points that match at the points, as well as having matching first and second derivatives at the points. In our problem, the points represent averages over a season; thus, we are more interested in finding a set of interpolating polynomials that match well at the boundaries, and that yield the proper seasonal averages when appropriately integrated. This philosophy is similar to the usual definition, so the method will still be referred to as spline interpolation.

Spline Interpolation: Introduction

The first spline method discussed will simply use linear functions in each season. That is, in each quarter, the function is assumed to have the form $b_i(t) = c_{0i} + c_{1i}t$. This gives eight unknowns. The seasonal averages provide four constraint equations, and the requirement that the various functions match at their respective boundaries provides another four. These constraints would appear to provide equations to determine the unknown, but the equations are not independent.

Linear Splines

Let the time argument t vary from -1 to $+1$ in each season (quarter). The seasonal averaging process (integrating each function over $[-1,1]$) yields the constant terms directly.

$$c_{0i} = 0.5 b_i. \quad (32)$$

The requirement of continuity across seasonal boundaries (including between seasons 4 and 1 at the year-to-year boundary) produces (after rearrangement and replacing c_{0i}).

$$c_{11} + c_{12} = 0.5(b_2 - b_1), \quad (33)$$

$$c_{12} + c_{13} = 0.5(b_3 - b_2) , \quad (34)$$

$$c_{13} + c_{14} = 0.5(b_4 - b_3) , \quad (35)$$

$$c_{14} + c_{11} = 0.5(b_1 - b_4) . \quad (36)$$

Unfortunately, these equations are inconsistent, as can be demonstrated by adding the first and third equations and subtracting the second and fourth equations. This leads to an unacceptable constraint on the data itself; that is:

$$0 = b_1 + b_3 - b_2 - b_4 . \quad (37)$$

We can solve problems of this type if we bring in an additional ad hoc requirement. For example, we may try to find the set of linear functions that has the least total variance while still exactly satisfying a subset of the original constraint equations, or we might look for the set that has the smallest overall coefficients in the fit. This, then, becomes a constrained least-squares problem and methods of solving them (particularly with the small numbers of equations involved) are no more difficult than normal least squares problems (Brandt, 1970; Claerbout, 1976; Golub, 1983; Meyer, 1982). In particular, if the absolute constraints we require are given by the matrix equation

$$Gc = d \quad (38)$$

where c is the column of coefficients we are looking for, and the goal is to minimize the energy of the coefficients (that is minimize the scalar $c^T c$), then the solution is given by:

$$c = G^T(GG^T)^{-1}d . \quad (39)$$

Cubic Splines

As above, we attempt to find a set of four cubic polynomials, one for each season, which match at the seasonal and annual boundaries and also have continuous first and second derivatives there. Further, we require that the seasonal averages computed from these functions match the four given values. As above, we start off appearing to have enough equations to solve for all 16 unknowns (four seasons, and four coefficients for each cubic polynomial): the four seasonal average equations and four equations each from continuity of the functions, their first and their second derivatives across boundaries.

In fact, again as above, the equations turn out to be linearly dependent, and some type of least-squares solution, constrained to meet the remaining independent equations exactly, must be performed. Rather than do this solution at this time, the requirement of continuity of the second derivative is dropped, and the order of the interpolating polynomials is dropped down to quadratic in the following section.

Quadratic Splines

After reading the previous two sections, one might wonder why another spline order should be attempted, but in this case it can be solved without external ad hoc requirements or least-squares criteria.

Each of the four seasons is represented by a unique quadratic function in t . For convenience, separately define the time to run from $t = -1$ to $t = +1$ for each season. Then the four functions each have the following form.

$$f_i(t) = c_{0i} + c_{1i}t + c_{2i}t^2, \quad (40)$$

$$f'_i(t) = c_{1i} + 2c_{2i}t. \quad (41)$$

The continuity of f across seasonal boundaries provides the first four equations.

$$f_1(+1) = f_2(-1) \rightarrow c_{01} + c_{11} + c_{21} = c_{02} - c_{12} + c_{22}, \quad (42)$$

$$f_2(+1) = f_3(-1) \rightarrow c_{02} + c_{12} + c_{22} = c_{03} - c_{13} + c_{23}, \quad (43)$$

$$f_3(+1) = f_4(-1) \rightarrow c_{03} + c_{13} + c_{23} = c_{04} - c_{14} + c_{24}, \quad (44)$$

$$f_4(+1) = f_1(-1) \rightarrow c_{04} + c_{14} + c_{24} = c_{01} - c_{11} + c_{21}, \quad (45)$$

Continuity of the first temporal derivative f' across seasonal boundaries implies that

$$f'_1(+1) = f'_2(-1) \rightarrow c_{11} + 2c_{21} = c_{12} - 2c_{22}, \quad (46)$$

$$f'_2(+1) = f'_3(-1) \rightarrow c_{12} + 2c_{22} = c_{13} - 2c_{23}, \quad (47)$$

$$f'_3(+1) = f'_4(-1) \rightarrow c_{13} + 2c_{23} = c_{14} - 2c_{24}, \quad (48)$$

$$f'_4(+1) = f'_1(-1) \rightarrow c_{14} + 2c_{24} = c_{11} - 2c_{21}. \quad (49)$$

The final four relationships among the 12 c_{ij} are derived using the seasonal averages. For each of the four seasons, there is a constraint that

$$\int_{-1}^{+1} f_i(t)dt = b_i, \quad (50)$$

where the b_i are the four given seasonal means. Performing the integration results in the four equations:

$$3c_{0i} + c_{2i} = 3b_i. \quad (51)$$

All of these relations can be compactly represented in matrix form:

$$\begin{pmatrix} 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} c_{01} \\ c_{11} \\ c_{21} \\ c_{02} \\ c_{12} \\ c_{22} \\ c_{03} \\ c_{13} \\ c_{23} \\ c_{04} \\ c_{14} \\ c_{24} \end{pmatrix} = \begin{pmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \\ 3b_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (52)$$

This matrix can be solved to yield

$$c_{01} = \frac{19}{16}b_1 - \frac{1}{8}b_2 + \frac{1}{16}b_3 - \frac{1}{8}b_4 \quad (53)$$

$$c_{11} = \frac{3}{8}(b_2 - b_4) \quad (54)$$

$$c_{21} = -\frac{9}{16}b_1 + \frac{3}{8}b_2 - \frac{3}{16}b_3 + \frac{3}{8}b_4 \quad (55)$$

for the first season. By direct solution, or by symmetry, the solutions for the other three seasons can be found by permuting the index to represent the season appropriately. For example, to solve for season 2, replace the second index of the c_{ij} by "2" and permute the subscripts on b from [1234, to [4123]. By continuing to permute circularly, the values of c_{i2} through c_{i4} are generated.

Summary Several methods permit quantities known only as seasonal values to be converted into continuous time series. These fields can be used as continuous forcing functions for models or as a continuous climatological relaxation term in the model equations. Errors in the seasonal values due to averaging and sampling (aliasing) were described. A companion report is being written that proposes a method for computing continuous time series that do not have these undesirable effects.

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A large-scale, integrated program in ocean circulation model development has been pursued by the Ocean Modeling and Prediction Branch, Ocean Sensing and Prediction Division, Naval Ocean Research and Development Activity. This program will provide the Navy with accurate nowcasts and forecasts of the state of the ocean on global, regional and tactical scales. One of the constraints that has been applied to this research is the development of circulation models that accurately reproduce the measured climatologies of such quantities as the dynamic height, thermocline depth, and depth-averaged density. These fields are derived using in situ data that have been acquired over many decades, but are so irregularly distributed that in most regions only annual or seasonal mean fields have been derived. Comparisons between model and data climatologies are facilitated if both are available as continuous time series, but the standard Levitus climatology is generally available at time scales no shorter than seasonal. This report describes several methods for converting the existing seasonal quantities into a continuous time series of fields, and also notes an amplitude error and bias present in the seasonal values as presently derived.

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